Gambling Is Bad

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Abstract

The subjective interpretation of probability posits the following tenets:

1) we have beliefs;

2) beliefs are in continuous[2] degrees: the so-called degrees of belief;

3) the degrees of belief of a rational agent are mathematically expressed by a probability function.

Mainstream Bayesianism develops 3 by the following assumptions:

3.1) the degrees of belief are measured by gambling in the form of betting quotients (or in the form of betting odds equivalently);

3.2) if betting quotients are assessed rationally, then they satisfy the axioms of probability and therefore they are probability functions.

In this paper (UPDATED ON 15/JUN/2008) I argue against 3.1 and 3.2, but I also outline an alternative way to support tenet 3.

Introduction

Gambling is one of mankind's oldest activities.

Discoveries of archeological relics such as writings, equipment and pictures have brought an uncontroversial evidence in support of this claim.

Dice games[3] were popular in all ancient civilizations across different cultures throughout the world.

In the Renaissance, the study of probability started to appear as a necessity in order to improve the chances of winning in gambling games[4].

In 1654 a French man, who has become world-wide known as the Chevalier de Méré, asked

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[2] This is not always the case: the (mainstream of) belief revision theories are dichotomous in dealing just with full belief and no belief (Hanson 2003).

[3] Julius Caesar said '*Iacta alea est*' ('The die is cast') when he crossed the river Rubicon in 49 BC.

[4] It's worth to mention the important book of the Italian Girolamo Cardano: *Liber de ludo aleae.*

the mathematician Blaise Pascal (1623-1662) to determine whether getting at least a six in 4 throws of one dice was more likely than getting at least a double six in 24 throws of two dice. Pascal exchanged several letters with the mathematician Pierre de Fermat (1601-1665) and they created the calculus of probability.

Reverend Thomas Bayes (1702-1761) showed that the same rules of the calculus of probability could also be exploited in problems of inverse inductive inferences: from the observed outcomes to the chance setup that produces the events.

Pierre-Simon de Laplace (1749-1827) re-formulated Bayes' theorem independently in 1774[5] and used it to provide a probabilistic estimate on the mass of Saturn:[6]

> '...it is a bet of 11,000 to 1 that the error in this result is not 1/100$^{\text{th}}$ of its value' (Laplace 1787)

As this quotation exemplifies, at that time people used expressions of gambling odds to talk about probability.

However, an explicit definition of probability in terms of gambling had to wait until the middle of 1920s, when more rigorous mathematical arguments became available.

In 1933[7], after almost three centuries from the conception of the calculus of probability, Andrei Nikolajevich Kolmogorov (1903-1987) gave the theory of probability an axiomatization, which has become quite orthodox by now.

The Kolmogorov axioms are based on the concept of a *sample space* σ, which is a set of elementary events. Then a *σ-algebra* Σ[8] is made on it, which is a non-empty collection of subsets of σ that is closed under complementation and countable unions of its members. Finally a *probability measure*, or *probability function*, p is a map from subsets $\mathcal{H}_i$ of Σ to the real numbers such that

1. (Non-negativity) $p\left(\mathcal{H}_i\right) \geq 0 \quad \forall \mathcal{H}_i \in \Sigma$
2. (Normalization) $p\left(\sigma\right) = 1$

[5] *Mémoire sur la probabilité des causes par les événements.*

[6] Source: Lewicki (2001). I remember to have read this statement and several similar statements, in Laplace's works (available in the *Oeuvres Completes de Laplace*, Gauthier-Villars) while preparing my dissertation (on celestial stability) for my degree in physics.

[7] *Grundbegriffe der Wahrscheinlichkeitrechnung.*

[8] If σ is a sample space with cardinality $\#(\sigma) = n \in \mathbb{N}$, then its power set 2^σ is a σ-algebra Σ and $\#(\Sigma) = 2^{\#(\sigma)} = 2^n$.

3. (Countable additivity)[9] $p\left(\bigcup_{i=1}^{\infty}\mathcal{H}_i\right)=\sum_{i=1}^{\infty}p\left(\mathcal{H}_i\right)$ whereas $\mathcal{H}_i$'s are countably infinite and pairwise disjoint.

If now (Hájek 2002, Earman 1992), as members of σ we consider sentences of a formal language[10] (closed under countable truth-functional combinations) instead of the elementary events, then we obtain the following related axiomatization:

1'. (Non-negativity) $p\left(\mathcal{H}_i\right)\geq 0 \quad \forall\mathcal{H}_i\in\Sigma$

2'. (Normalization) $p\left(\top\right)=1$ whereas $\top$ is a tautology in classical logic

3'. (Countable additivity)[11] $p\left(\bigvee_{i=1}^{\infty}\mathcal{H}_i\right)=\sum_{i=1}^{\infty}p\left(\mathcal{H}_i\right)$ whereas $\mathcal{H}_i$'s are countably infinite and pairwise logically incompatible.

The triple (σ,Σ,p) is called *probability space.*

Up to the time of the axiomatization by Kolmogorov, the probability was mainly interpreted as a logical objective feature of the world. Eventually a lot of difficulties and paradoxes were associated with that kind of interpretation – problems arising from the Keynes' principle of indifference in the first place[12] – and the need for a different interpretation of probability became apparent.

Gambling is for the man in the street

The *subjective interpretation* of probability was conceived by the English Frank Plumpton Ramsey (1903-1930) and by the Italian Bruno de Finetti (1906-1985) in the middle of the 1920s independently.

According to the subjective interpretation of probability, probabilities are nothing but the representation of the subjective degrees of belief[13] in the (conclusive) verification of hypotheses, be they events or propositions. Probabilities interpreted in this way are called

[9] De Finetti, one the greatest opponent to countable additivity, opted for a finite additivity $p\left(\mathcal{H}_i\cup\mathcal{H}_l\right)=p\left(\mathcal{H}_i\right)+p\left(\mathcal{H}_l\right) \quad \forall i,l$ (with $i\neq l$) : $\mathcal{H}_i\cap\mathcal{H}_l=\emptyset$. Anyway the analysis of this difference is not strictly necessary now.

[10] Warning: the translation of axiomatization from the mathematical one to the Bayesian one is more delicate than how it is presented here. See, e.g., (Earman 1992).

[11] Finite additivity here would translate to $p\left(\mathcal{H}_i\vee\mathcal{H}_l\right)=p\left(\mathcal{H}_i\right)+p\left(\mathcal{H}_l\right) \quad \forall i,l$ (with $i\neq l$) : $\mathcal{H}_i\wedge\mathcal{H}_l\models\bot$.

[12] Ronald Aylmer Fisher (1890-1962) wrote about that principle: 'leads to apparent mathematical contradictions. In explaining these contradictions away, advocates of inverse probability [this was the traditional name for the use of Bayes' theorem to generate posterior probabilities] seem forced to regard mathematical probability ... as measuring merely psychological tendencies, theorems respecting which are useless for scientific purposes'. Source: Howson & Urbach (1989).

[13] Degrees of belief are also called credences.

subjective probabilities. As de Finetti writes:[14]

> 'In the conception we follow and sustain here, only subjective probabilities exist – i.e., the degree of belief in the occurrence of an event attributed by a given person at a given instant and with a given set of information.' (de Finetti 1950)

or, in another of his books: probability[15]

> 'means degree of belief (as actually held by someone, on the ground of his whole knowledge, experience, information) regarding the truth of a sentence, or event E (a fully specified 'single' event or sentence, whose truth or falsity is, for whatever reason, unknown to the person).' (de Finetti 1968)

So, for example, if a person assigns probability $p = \frac{1}{2}$ to a coin for landing on heads, it means that *that person* is equally confident between heads or tails. The subjective interpretation of probability is also called the *personalist interpretation* of probability.

From this philosophical commitment, as the mathematical probability is a quantity expressed by a real number between 0 and 1 (extremes included), and probability means degrees of belief, then it follows that also degrees of believe are expressed by a number between 0 and 1 (extremes included). The problem here is how to justify this argument. The orthodox solution to this problem consists in measuring the personal degrees of belief and showing they have all the properties of probabilities.
But how to measure degrees of belief?

Ramsey, in his 'Truth and probability', makes an interesting consideration in this regard:[16]

> 'It is, I suppose, conceivable that degrees of belief could be measured by a psychogalvanometer or some such instrument' (Ramsey 1926)

[14]Source: Nau (2001).
[15]Source: Galavotti (2001).
[16]Until new specification, the source of the quotations is Gillies (2000).

Now I present a few personal thoughts about this sentence.

First of all, mathematical probabilities cannot be logically inconsistent. Then are we too sure to cannot have inconsistent beliefs? Well, if believing is an action (as it is), it would seem that indeed we cannot carry out actions that are logically inconsistent.

On the other hand we are not talking of the action of believing *per se*, but of the content of that action: that is what we call beliefs and degrees of belief.

But, where is this content of believing, in order to measure it?

The content of the thought would seem to be at an ontological level different from the act of thinking. However, if we try to identify that level with the conscious one (having arguments like the Chalmers' zombies in our support), then we'd obtain that the degrees of belief cannot be measured by any physical instrument (as they are of a different metaphysical substance), contradicting Ramsey's words.

If, on the other hand, we take that the content of the thought has just a physical support[17], or it has an identity to a physical support, then the degrees of belief would seem to be, at least in principle, measurable by some physical instrument; in which case, the content of believing would be detected as a physical signal, and I cannot conceive how a physical signal can be not consistent[18].

Moreover I see the following as a further riddle: from the way we act we infer that we have certain beliefs about the world more than others. For example, when we walk we believe that the ground will be under our foot also at the the next step and at a certain distance from our foot. As the content of the thought is made also of propositions, we can think of an actual infinity of possible propositions concerning our representation of the world. By an example different from the general issue of logical omniscience: we can believe 'the distance of that point of the ground from our foot is 0.51 m' is probable, and 'the distance of that other point of the ground from our foot is 0.52 m' is not very

[17] In Ramsey's sentence there is an implicit assumption of physicalism; namely, all the existing facts of the world are physical.

[18] And I must admit also that I cannot either conceive how a mental degree of belief we have at one instant is inconsistent with the others: at any given time we are thinking of some content of thought or not. I don't understand how we can think of something and think of not that something (with incompatible degree) at the same time. In other words, I have difficulties at understanding the request of logical consistency (i.e., probabilistic coherence) for a *synchronic* quantification of the degrees of belief.

probable, and so on. But we must also acknowledge that there are infinite points, infinite distances, infinite theoretical objects and so on, and therefore, in a physicalist context we should have an actual infinity of degrees of beliefs that are available for being measured in the brain. But, as this infinity of information would normally request an infinite material support, then this would be contrasting with the physicalist assumption that the brain is finite. Therefore Ramsey's words would be contradicted again.

However, a few pages later Ramsey withdraws the possibility of a physical measurement:

> 'it seems to me observably false, for the beliefs which we hold most strongly are often accompanied by practically no feeling at all; no one feels strongly about things he takes for granted.' (Ramsey 1926)

Gillies comments in support of Ramsey:

> 'When I cut a slice of bread to eat, I believe very strongly that it will nourish rather than poison me, but this belief, under normal circumstances, is not accompanied by any strong feelings, or indeed any feelings at all'. (Gillies 2000)

But I don't think these arguments are sound: in fact, in the physicalist assumption of measurability of the degrees of belief, then, if something is believed 'very' or 'most strongly' (or less strongly), it must be somewhere out there in the brain. Therefore, even if a belief is not accompanied by any feelings at all, since it still is something of mental, then it must be measurable somehow by a physical instrument[19].

A possible remedy to my argument of "the infinity of the mind Vs the finiteness of the brain" would be to assume that, in our representations of the world, we contemplate only a finite number of aspects, and therefore, there is only a finite number of beliefs and the mind too is finite.

Still, the measurability of the degrees of belief is not obvious: for example, could we then

[19] Quantum effects could be quantitatively relevant to this issue, but not for the most interesting qualitative soundness of the argument.

measure the degrees of belief to a dog, or to a mosquito, or to a bacterium, or to a protein? A positive answer would seem not acceptable. A negative answer, on the other hand, would lead to the following question: where is the demarcation among different beings (man and woman included)? Now, a demarcation can be neat or not neat. A neat demarcation would appear conterintuitive as there is a continuity in the mental capabilities of those beings. A not neat demarcation would seem counterintuitive too, as measurability is associated with a metaphysical possibility: either it is neatly possible[20] to measure something or it is not.

Finally I believe that a physical instrument capable of measuring our degrees of beliefs is a secondary aspect: if there was a science-fiction machine that displays what our degrees of belief are, if we feel that those values are not believable, then we don't believe that machine "reads" our mind: the first and most authoritative *measurement* of our degrees of belief is made by us *subjectively*. If there was no material world and no other people, still *we* would be able to distinguish that we have *different* degree of belief.

Then, having discarded the measurement by a physical instrument, Ramsey gets to conclude what has become a major assumption of Bayesianism:

> 'The old-established way of measuring a person's belief is to purpose a bet, and see what are the lowest odds which he will accept. This method I regard as fundamentally sound.' (Ramsey 1926)

After a few pages he gives a sort of justification:

> '. . . this section . . . is based fundamentally on betting, but this will not seem unreasonable when it is seen that all our lives we are in a sense betting. Whenever we go to the station we are betting that a train will really run, and if we had not a sufficient degree of belief in this we should decline the bet and stay at home.' (Ramsey 1926)

Nevertheless, Ramsey insisted on an operational definition of probability, that is to say, based on an observable method of measuring it.

[20]At least in principle.

De Finetti too proposed that the degrees of belief can be measured within a betting scenario, but he did not look at it as a general method:[21]

> 'because the person may have a special eagerness or reluctance to bet.' (de Finetti 1962a)

Therefore, allegedly for the first time in the history, de Finetti explicitly advances a definition of probability:

> 'the psychological sensation of an individual' (de Finetti 1931a)

which is different from the (inter-subjective) method to measure it[22].

De Finetti reasonably seemed to observe also that the act of measurement alters the value of the quantity to measure and so, the result of the measurement cannot be genuinously representative. In fact, the result of the measurement cannot be not synchronic with the sensation and the act of measurement influences a change in that sensation.

I think de Finetti is certainly right in taking probability as the instanteous 'psychological sensation' instead of the result of some procedure. Indeed, this sensation existed even before mankind conceived the concept of probability or used any kind of gambling to measure it. Moreover, we can easily imagine of experiencing a phenomenology, according to which we would say to be in hell or in an OBE[23], where (physicalism would fail and) any measurement is impossible, but still we would have the psychological sensation of probability.

Besides, de Finetti worked out also several other (than bets) equivalent estimation methods to calibrate one's own "psychological sensation" carefully: the method of the *proper scoring rules* (Howson & Urbach 1989) (Earman 1992) has become the most famous one, for which de Finetti made experiments with his students to test its goodness.

Therefore, considering the probability as meaningful only when measured in a scenario of bets, or some other gambling, is a conceptual mistake.

[21] Source: Galavotti (2001).

[22] Von Mises, pioneer of the frequentist interpretation of probability writes in 1928: 'The relative frequency of the repetition is the 'measure' of probability, just as the lenght of a column of mercury is the 'measure' of temperature. Source: Gillies (2000).

[23] Out of Body Experience.

Hereafter I'll use the letter p to denote the subjective probability (as a psychological degree of belief) and q to denote the value that measurement of it has returned.

I conclude this section by the words of professor Maria Carla Galavotti[24]:

> 'The idea that probability can be defined in various ways is a central feature of de Finetti's perspective, where the scheme of bets represents only a convenient device for talking about probability in a way that makes it understandable to the "man in the street".' (Galavotti 2001)

The Dutch stuff

The term 'Dutch book' is always invoked in any discourse on the subjective interpretation of probability or on the foundations of Bayesianism in particular; the literature related to the Dutch book is almost infinite.

On the other hand, 'Dutch book'[25] is often associated with four different concepts (Hájek 2005): the Dutch book, the Dutch-book theorem, the converse Dutch-book theorem and the Dutch-book argument.

A *Dutch book* is a group of bets such that, if an agent accepts to play, there is a mathematical certainty of losing.

The *Dutch-book theorem* would consist in proving that if the agent's subjective degrees of belief don't satisfy the axioms of probability, then there is a possibility of making a Dutch book against her.

The *converse Dutch-book theorem* would guarantee that if the agent's subjective degrees of belief do satisfy the axioms of probability, then there is not a possibility of making a Dutch book against her.

The *Dutch-book argument* aims to prove that the (subjective) degrees of belief of a "rational" agent obey the calculus of probability.

I will now bring an example[26] of how the Dutch-book theorem works, but, first of all,

[24] Whom I know personally; she has written numerous works about Ramsey and de Finetti.

[25] I've spent several days to find the rationale of the ethnical etymology of this expression coined by Ramsey, but with no success. On the other hand I've found out that, eighty years after the birth of the 'Dutch book' (in 1926), there is no agreement on the etymology (see (Hacking 2001) or (Wakker 2001)).

[26] This simple example is completely original.

a short clarification of the used terms is needed.

Suppose an *a*gent A bets a_j € *on* the (conclusive) verification of $\mathcal{H}_j$ and a *b*ookie B bets b_j € *against* the (conclusive) verification of $\mathcal{H}_j$. The sum $S_j = a_j + b_j$ € is called *stake.*

If $\mathcal{H}_j$ is verified then A collects the stake and gets a net gain of b_j €.

Otherwise B collects the stake and A gets a net gain of -a_j €.

The proportion $q_j = \frac{a_j}{a_j+b_j}$ is called A's *betting quotient* on $\mathcal{H}_j$.

I can summarize what I've just said by the following payoff matrix of `Figure 1`.

	A pays	A collects	A's net gain
$\mathcal{H}_j$ is verified	$q_j S_j$ €	S_j €	$(1-q_j) S_j$ €
$\mathcal{H}_j$ is not	$q_j S_j$ €	0 €	$-q_j S_j$ €

`Figure 1`

Real-life gamblers don't talk about betting quotient, but just about *odds.* If an agent bets a_j € hoping to make a profit of b_j € from the verification of $\mathcal{H}_j$, then gamblers say that the odds are a_j *to* b_j on[27] $\mathcal{H}_j$, in symbols: $\mathcal{O}_j = a_j/b_j$. Therefore the following relations links the odds with the betting quotients: $q_j = \frac{\mathcal{O}_j}{1+\mathcal{O}_j}$ and $\mathcal{O}_j = \frac{q_j}{1-q_j}$.

Now a slightly different problem for another understanding a new definition: after having fixed the stake (for example $S_j = 1$ €), the agent A must choose a single betting quotient q_j such that she is indifferent between

betting on $\mathcal{H}_j$ at betting quotient q_j for winning $(1 - q_j)S_j = b_j$ € if $\mathcal{H}_j$ is verified; or

betting against $\mathcal{H}_j$[28] at betting quotient $1 - q_j$ for winning a_j € if $\mathcal{H}_j$ is not verified.

Such a betting quotient q_j is called a *fair betting quotient* and leads to a *fair bet*[29].

So, contrary to what is done by bookies in real-life situations, when fair betting quotients occur, then there is no "edge"[30] among the quotients and the agent believes that she has *no advantage* in betting on $\mathcal{H}_j$ at rate q_j or betting on $\neg\mathcal{H}_j$ at rate $(1 - q_j)$. If a person has no advantage in playing as an agent or as a bookie, then the expected net gain

$$\mathbb{EU}(\mathcal{H}_j) = b_j \cdot q_j - a_j \cdot (1 - q_j)$$

[27]And the odds against $\mathcal{H}_j$ are b_j/a_j.

[28]Or, in other words, betting on $\neg\mathcal{H}_j$.

[29]Another analogous concept is the one of *fair price*: the (single) price q_j € for which an agent would be indifferent between buying or selling the access a bet where the payoff is 1€ if $\mathcal{H}_j$ is verified, and nothing otherwise (Hájek 2002).

[30](Hacking 2001).

is zero (euros). It is straightforward to test that if $q_j = \frac{a_j}{a_j+b_j}$, then $\mathbb{EU}(\mathcal{H}_j) = 0$€[31].

Let's see what happens in case of "edge", that is, when the sum of the betting quotients for playing on and against the same hypothesis is different than 1.

Suppose, for instance, that our agent is a bit silly and she accepts to offer a $q_1(\mathcal{H}_1) = 0.51$ and a $q_2(\mathcal{H}_2 = \neg\mathcal{H}_1) = 0.52$ (as she sillily thinks they are fair betting quotients). Then the game consists in letting the bookie to decide the stakes of the two bets (in the general case, though, the bets can be more than two or just one). Moreover, each S_i can be taken *positive or negative* depending on what the bookie's interest is; this condition of the Dutch book, where the bookie chooses the stake (instead of where the offered betting quotients can only be positive, the stakes are fixed and the bookie decides whether the agent must sell or buy), correspond to the equivalent formulation held by Carnap[32] and de Finetti eventually, and it is the one presented in (Gillies 2000)).

Now, the cunning Dutch bookie B chooses a single common stake $S = S_1 = S_2 =$100€, and so asks the agent to bet $a_1 = 51$€ on $\mathcal{H}_1$ and $a_2 = 52$€ on $\mathcal{H}_2$.

So, according to the bet 1: A's net gain is +49€ if $\mathcal{H}_1$ is verified and -51€ if $\mathcal{H}_1$ does not. On the other hand, according to the bet 2: A's net gain is -52€ if $\mathcal{H}_2$ is verified and +48€ if $\mathcal{H}_2$ does not. But, here, $\mathcal{H}_2 = \neg\mathcal{H}_1$. Therefore, if $\mathcal{H}_1$ is verified then A's overall net gain is -3€, while if $\mathcal{H}_2$ is verified then A's overall net gain is -3€ again: no matter what the destiny of $\mathcal{H}_1$ or $\mathcal{H}_2$ is, our agent A is bound to lose 3€!

This is an example of a Dutch book. A Dutch book is also called a *sure-loss contract.*

It is also an example of Dutch-book theorem: the agent's degrees of belief – measured by the betting quotients – didn't satisfy the axioms of probability and it's been possible to make a Dutch book against her. The example of the converse Dutch-book theorem is straightforward: if she chose the betting quotients so that $q_1 + q_2 = p(\mathcal{H}_1) + p(\mathcal{H}_2) =$

[31] Notably, Ramsey developed an approach based on utilities instead of monetary values. However, the utilities may numerically coincide with the monetary values and they must face additional major problems such as the ascertainability of the values of utilities. I present a simplified version: calling $\mathbb{U}(\mathcal{H}_j)$ (or $\mathbb{U}(\neg\mathcal{H}_j)$ or $\mathbb{U}(\mathcal{H}_i)$) the utility associated with verification of $\mathcal{H}_j$ (or $\neg\mathcal{H}_j$ or $\mathcal{H}_i$), and supposing that the agent is indifferent between $\mathbb{U}(\mathcal{H}_i)$ or the gamble '$\mathbb{U}(\mathcal{H}_j)$ if $\mathcal{H}_j$ is verified or $\mathbb{U}(\neg\mathcal{H}_j)$ if $\neg\mathcal{H}_j$ is verified', then the degree of belief in $\mathcal{H}_j$ is defined as $p(\mathcal{H}_j) = \frac{\mathbb{U}(\mathcal{H}_i)-\mathbb{U}(\neg\mathcal{H}_j)}{\mathbb{U}(\mathcal{H}_j)-\mathbb{U}(\neg\mathcal{H}_j)}$. An easy justification of this definition comes from observing that the agent's indifference is for $\mathbb{U}(\mathcal{H}_i) = \mathbb{U}(\mathcal{H}_j)\ p(\mathcal{H}_j) + \mathbb{U}(\neg\mathcal{H}_j)\ p(\neg\mathcal{H}_j)$ and $p(\neg\mathcal{H}_j)$ is $1 - p(\mathcal{H}_j)$.

[32] (Heilig 1978).

$p(\mathcal{H}_1) + p(\neg\mathcal{H}_1) = p(\mathcal{H}_1 \vee \neg\mathcal{H}_1) = 1$, for instance by choosing $q_2 = 0.49$, then two easy calculation brings to the conclusion that our agent would not have lost any euro cent. A group of betting quotients, such that no Dutch book can be made on it, is called *coherent*. This idea of coherence is somehow the inductive-logic counterpart of the idea of consistency in deductive logic[33]. Rigorous proofs of the Dutch-book theorem and its converse can be found in Kemeny (1955), Lehman (1955) and Skirms (1986)[34]. Following Gillies (2000), it's now possible to merge the two theorems in one *Ramsey-de Finetti theorem* which states: a group of betting quotients is coherent *if and only if* they satisfy the axioms of probability.

Finally the Dutch-book argument. As the example of our silly agent has showed:

Premise I) if she had been rational then she would have offered a coherent[35] group of betting quotients (else sure loss)

Premise II) a coherent group of betting quotients satisfy the axioms of probability (*Ramsey-de Finetti theorem*)

Conclusion) rationality leads to offer betting quotients that *always* satisfy the axioms of probability (in order to rationally avoid sure losses).

For the purpose of a better understanding of the "Dutch stuff" and for the reference in next sections, I present the short proof of the Ramsey-de Finetti theorem for the only Kolmogorov axiom 2' as suggested in Gillies (2000).

Coherence $\Rightarrow$ Axiom 2'.

If our agent chooses $q_j(\top) > 1$ (or < 1), then the bookie wins by choosing $S_j > 0$ (or < 0). Therefore the only case $q_j(\top) = 1$ is remained.

Axiom 2' $\Rightarrow$ Coherence.

If our agent chooses $q_j(\top) = 1$, then it's not possible for the bookie to 'win, since the stake, whatever its sign, is simply passed from one to the other and then back again.'

Now we can pass to criticize how the Dutch-book argument is used in discussions con-

[33]In other words, probabilistic coherence is the counterpart of logic consistency.

[34]Source: Hájek (2005).

[35]Probabilistic coherence is taken to be necessary for rationality. The same is also a sufficient condition for rationality according to *subjective Bayesianism*. *Objective Bayesianism* holds this condition is not sufficient *per se*, but that the degrees of belief are uniquely fixed among agents, who have the same background knowledge, by means of empirical and logical constraints (Williamson 2005).

cerning degrees of belief.

Dutch-booking is bad

The Bayesian interpretation of probability holds that the degrees of belief are analyzed in terms of gambling behavior. Two sections ago we saw that this is not essential, and is also *a priori* troublesome. Moreover, Bayesianism holds that the agent's rationality *per se* is sufficient for providing a set of betting quotients that satisfy the axioms of probability. Let's see, a bit more in detail, why such "rational" betting quotients can*not* be a good measure for the degrees of belief and can*not* generally work as a probability function either.

Here I list some examples of objections that have been raised against the Dutch-book argument. The ones without reference[36] are original (unless some one is guilty of anticipatory plagiarism).

1] The actual verification of the rationality of the agent's degrees of belief requires an environment with other people, which is conceivable of being not always available (this point has been raised by many authors, though with a different aim).

2] Expressing the agent's beliefs and carrying out a bet requires a non-infinitesimal time. The agent could change her degrees of belief in the meantime. Therefore the Dutch book should always be considered only in a more general version: diacronically and not synchronically. (The criticism of this paper is just focused against the standard way of invoking the Dutch-book argument, namely synchronically.) However, also the diacronic Dutch-Book argument seems to be 'fallacious' (Maher 1993) as many cases show when considering possibilities like mental illness or drugs (e.g., see (Bradley & Leitgeb 2006)).

3] Suppose the agent has very little money and $\mathcal{H}_j$ seems a tautology to her. If she accepts to bet, then (see `Figure 1`) she has no net gain. So why should she play? She has no possibility of gaining anything. Besides, if she has an infinitesimal possibility of being

[36] A few of the ones with reference were not used against the Dutch-book argument, but they inspired me to do so explicitly.

wrong, then she has an infinitesimal possibility of losing, but none of winning[37]! Similarly this could be said for a contradiction.

4] Suppose the agent has very little/very much money, then she is not so interested in providing good q_i's because she doesn't care of risking/wasting some money. Besides, there is no neat demarcation between very little/much and not very little/not very much. Therefore, the choice of the agent couldn't be (as it is) considered independently of the quantity of all her money. (de Finetti 1962a)

5] The last two points are for a Dutch book *à la* de Finetti, with monetary values, but they can apply also for a Dutch book *à la* Ramsey by replacing 'money' with 'utility'.

6] In the usual case where there are no contradictions or tautologies to bet on, then: for *any* group of bets, if the agent *always* assigns betting quotient 1 to the first hypothesis $\mathcal{H}_1$ and 0 to all others[38], then she can*not* be always Dutch-booked. This sounds paradoxical. Let's look at an example. The group of bet consists in having tomorrow: $\mathcal{H}_1$ as a sunny, $\mathcal{H}_2$ as a rainy day, $\mathcal{H}_2$ as a foggy day, and so on. How would the bookie choose the stakes here? According to what the objective probabilities are! But, first of all, this should be a problem independent from the Dutch-book argument (i.e., the existence of objectivity). Secondly, de Finetti started his *Theory of probability*[39] by saying '[OBJECTIVE] PROBABILITY DOES NOT EXIST'. Moreover it might also turn out that the agent's choice is always winning accidentally. Clearly, though, the betting quotients can *not at all* reflect the subjective degrees of belief.

7] Suppose the agent has only 51 €, and she has to fly back to Italy, but the airplane ticket costs 100 €. Then a wicked (probably Dutch) bookie tells her that, if she gives him her 51 €, he will toss a coin: if heads he will give her 100 €, otherwise he will give her nothing. Here, since she needs to accept to play, $q_1(\text{heads}) = \frac{a_1}{S_1} = \frac{51}{100}$ while $p(\text{heads}) = \frac{50}{100}$. Again the betting quotients do not measure the degree of belief correctly. (Maher 1993)

8] Suppose the agent has to give the betting quotient for $\mathcal{H}_1$. Here either the agent believes that the bookie will choose $S_j > 0$ or $S_j < 0$. Let's call $\mathcal{H}_2$ what the agent actually

[37]This fact is analogous to the mathematical idea of unstable equilibrium.

[38]For simplicity of language, let's assume to consider just a partition of hypotheses here.

[39]de Finetti, B 1974, *Theory of probability: a critical introductory treatment*, Wiley, New York.

believes. If in the first case ($S_j > 0$), she has to offer $q_j = 0$ (see `Figure 1`). If in the second case ($S_j < 0$) then she has to offer $q_j = 1$. This, anyway, even *independently* of $p(\mathcal{H}_1)$! To check the rationality of $\mathcal{H}_2$, on the other hand, we would need a new test, but then a new $\mathcal{H}_3$ would appear, and so on *ad infinitum.* (Rowbottom forthcoming)

9] If $\mathcal{H}_j$ is a universal scientific hypothesis, then it can never be (conclusively) verified. Therefore any bet with $\mathcal{H}_j$ is never paid off and there is, again, a reluctance to enter in it. (Earman 1992)

10] The betting scheme seems to be rather unnatural if we imagine that a scientist has to put himself in gambler's shoes to evaluate the probability of his scientific hypothesis. (Zoccante 2005)

11] The agent must have perfect knowledge of how the conditions of the bet depend on the verification of $\mathcal{H}_j$: suppose that a dice is thrown (where the agent cannot verify directly), if the number is even then the agent is offered a bet on $\mathcal{H}_j$='The *resulted* number is odd' with a payoff in real euros, otherwise, if the number is odd she is offered a bet on $\mathcal{H}_j$ with a payoff in fake euros. Here, clearly, the agent wouldn't want to accept to enter in the bet by any q, even if her $p(\mathcal{H}_j)$ *is* $\frac{1}{2}$. (Bradley & Leitgeb 2006)

12] It is implicitly assumed that the bookie perfectly succeeds all the times to choose what is unfavorable for the agent, otherwise, whenever the agent does not satisfy the axioms of probability, he could even give her the certainty of winning: by choosing the opposite sign of the stakes with respect to what he normally (and nastily) does. (Hájek 2005)

13] Suppose the agent must bet on $\mathcal{H}_j$='I will die after the bet, but before I can collect the stake'. If $\mathcal{H}_j$ is verified, then she doesn't collect the stake, if $\mathcal{H}_j$ does not verify, then she doesn't collect the stake either. So, this bet is not a sure loss as long as it costs nothing: $q_j = 0$. But, unfortunately, the probability of dying is higher than 0. (Eagle 2006)

14] What stakes should the bookie choose for an hypothesis like $\mathcal{H}_1$='Julius Caesar has winked ten times crossing the Rubicon'? Would any value between 0 and 1 be rational? If so, even for $\mathcal{H}_2$='Julius Caesar has winked ten billion times crossing the Rubicon'? (Jackson & Pargetter 1975)

15] What is the role of the bookie choose for a hypothesis that can only be verified by the agent? (Howson & Urbach 1989)

16] Even when considering people in a casino, which is *in* a plain context of gambling, from their behavior it's possible to infer that they do *not* have a "rational" evaluation of the probabilities (or else, if they are not masochist, they wouldn't play). (Zoccante 2005)

17] All logical tautologies are infinite. Therefore the Dutch-book argument presupposes full logical omniscience[40]. (Strevens 2006)

18] Suppose the agent has decided to sell her car and all her material comforts in order to have less possibilities of recreation or loss of attention, and, by that, to focus on philosophy exclusively. She decided this "rationally"[41]. Therefore she offers all the betting quotients equal to 27 (her age) in order to fulfil herself as soon as possible. In this case the agent's rationality does not lead to a set of betting quotient that satisfy the axioms of probability. On the other hand, though, this conclusion has followed since, what is rational according to the agent, is not according to the bookie: the bookie sees that what she has decided "rationally" is not rational. Anyway, the point is that also the idea of rational agent is not trivial or unproblematic: its relativization would relativize the conclusion of the Dutch-book argument.

19] Suppose the agent has just been left by her boyfriend and the the agent must bet on $\mathcal{H}_j$='Your boyfriend will come back to you'. Now the agent is willing to bet any amount on a betting quotient equal to 0, because if $\mathcal{H}_j$ will be verified she would win a lot of money, if $\mathcal{H}_j$ will not be verified then she would be even happier for having her boyfriend back. Also in this case, betting quotients has nothing to do with her degree of belief in $\mathcal{H}_j$ (which may well tend to be very high).

20] Suppose the bookie asks the agents the betting quotient for a tautology $\mathcal{H}_j = \top$. Suppose also that the agent really wants to respect the probability axioms objectively. However she believs that $\mathcal{H}_j$ is a contradiction and therefore she offers $q_j = 0$ in order to comply with the probability axioms. She has wanted to follow the probability axioms

[40]Moreover, if the agents was logically omniscient, then she should know the logical entailments of *any* hypothesis by the previously known evidence and therefore, no new hypothesis would provoke a change in belief on the same evidence. However this would preclude actual cases where *old evidence* increases the agent's confidence in a new hypothesis. (Garber 1983)

[41]If we can say that exclusively focusing on philosophy is rational ... but this is the issue!

rationally in order to not lose money, but she lost necessarily!

21] The agent is told she will be shot if her betting quotients don't satisfy the axioms of probability.

This simple point is certainly the most interesting to me. The expressions of this sentence is taken from Maher (1993). Maher says that the 'argument has no force at all' for it could be used to show that her betting quotients don't satisfy the axioms of probability (or else she will be shot). But doesn't the Dutch-book argument make use of a similar threat[42]?

Let's imagine that Mr. C felt to quantify the strength of his degrees of belief from 0 to 2 (instead of to 1). There seems to be no conceptual counter-argument against this choice[43] (other than a loss of mathematical beauty). Besides, also Mr. C accepts that rational betting quotients have to satisfy the axioms of probability and it would be irrational to offer betting quotients otherwise.

But then, what is the relation between this Dutch-book argument and Mr. C's degrees of belief?!?

Surely, as Mr. C's degrees of belief cannot satisfy the axioms of probability, the Dutch-book argument does *not* show that Mr. C's degrees of belief satisfy the axioms of probability! Moreover, Mr. C finds no problems at all about this.

At most, *if* Mr. C's degrees of belief are "believed" to behave like a probability function, then, they can be normalized to 1 and they satisfy the axioms of probability like a group of rational betting quotients[45] does. But, having this common property (such as the normalization) doesn't mean having also all the remaining properties in common! On the other hand, the confusion between the system of degrees of belief and the system of the betting quotients is easy to commit, for, when we offer betting quotients, we have used also degrees of belief (although *not* vice versa). But there is no guarantee of a rigid linearity or fixed relations between the first system and the second one. Even if we *normed* that the "rational" degrees of belief have to be built like rational betting quotients, still we wouldn't shed any light on the degrees of belief: we *already assumed* them (first assump-

[42]As Maher concludes that finally it does.

[43]In fact, a good interpretation of the various *preferences* can be still done (see[44] the *representation theorem* for Bayesian decision theory).

[45]But also like a lot of other things that can satisfy the axioms of probability!

tion) to have the properties of probabilities (e.g., adding up to 1, etc.) as an *essential* condition. In fact, we decided to measure them (*before* measuring them) by betting quotients, and *not* by betting odds, right because the betting odds don't have the properties of a probability function. Accordingly we consider "good" or "rational" degrees of belief that keep on behaving as probability.

Moreover, the Dutch-book argument does not talk about how the degrees of belief should be in order to receive more conditions from the infinite possible functions satisfying that essential condition. And whether the Dutch-book argument would show that the "rational" degrees of belief satisfy the axioms of probability, it would do so *only* after having assumed that our degrees of belief exactly correspond to the betting quotients. But the Dutch-book argument does not justify this second assumption either.

So, is the Dutch-book argument analogous to having a gun put to our head?

I think it's worse than that!

Firstly, in fact, if we were in the contingent situation (but not less contingent than the usual one) where the bookie chose the opposite stakes to what he does normally, then we would conclude that: *rationality* leads to offer betting quotients that *never* satisfy the axioms of probability (that is, for surviving as well as for getting the stakes of the bookie we would be largely conditioned in what we say). Therefore, betting quotients do not represent degrees of belief.

Secondly, and this is the additional fact that makes it worse: it makes people to fallaciously identify degrees of belief (as a psychological sensation) with betting quotients (as a particular mathematical object) only because, presumably, they share a common mathematical property and because, often they appear in comparable strength, but, anyway, without giving any kind of explicit justification for this presumption.

As the co-founder of the subjectivist interpretation of probability puts it:[46]

> '...if you want to apply mathematics, you must act as though the measured magnitudes have precise values. This fiction is very fruitful, as everybody knows; the fact that it is only a fiction does not diminish its value as long as

[46] Source: Gillies (2000).

we bear in mind that the precision of the result will be what it will be... To go, with the valid help of mathematics, from approximate conclusions, I must go by way of an exact algorithm, even though I consider it an artifice.' (de Finetti 1931)

In conclusion, the usual way of appealing to the Dutch-book argument[47] is inessential (for it is useless, once scanned carefully), definitely misleading (as the huge literature on it proves), and so it is just bad.

Finally I outline three more original arguments which further enlarge the number of difficulties that the Dutch-book argument ought to overcome.

The Dutch bookie and the Tibetan agent

Suppose there is a Dutch bookie as before, but now there's a Tibetan man as an agent. Then the dreadful Dutch bookie posts the following problem:

> 'What are the betting quotients for the statements $\mathcal{H}_1$='Everything is real and not real' and $\mathcal{H}_2$='Everything is real or not real"?'

An orthodox analysis is to attribute a betting quotient $q_1 = 0$ to the first statement and betting quotient $q_2 = 1$ to the second statement. In fact the first one, $\mathcal{H}_1$, is a contradiction (and the law of contradiction is the most untouchable principle of logic). The second one, $\mathcal{H}_2$, is nothing but the Aristotelian law of the excluded middle (or *tertium non datur*):[48] $\mathcal{H}_3 \vee \neg\mathcal{H}_3$ is a tautology $\top$, i.e., a statement that is true in every interpretation (or model, or valuation). Therefore, *in classical logic*, the first statement $\mathcal{H}_1$ is the negation of the second statement $\mathcal{H}_2$ (which is contradiction $\perp$ instead): $\neg\mathcal{H}_2 = \neg(\mathcal{H}_3 \vee \neg\mathcal{H}_3) = (\neg\mathcal{H}_3 \wedge \neg\neg\mathcal{H}_3) = (\neg\mathcal{H}_3 \wedge \mathcal{H}_3) = (\mathcal{H}_3 \wedge \neg\mathcal{H}_3) = \mathcal{H}_1$.

[47] Another essay on its own should be dedicated to examine how the Dutch-book argument is used in conditional bets, in the diachronic Dutch-book argument, and how the Dutch-book argument is exploited in order to (try to) show that it is irrational to not revise one's subjective probability by means of the *Bayesian conditionalization rules*. However, many points of the present criticism apply also to those Dutch-book arguments.

[48] Let's use $\mathcal{H}_3$ to abbreviate the classical statement $\forall x\, \mathcal{H}_3(x)$.

But, now suppose further that, differently from ordinary circumstances:

I} the Dutch bookie is a supporter of intuitionist logic[49] (as many Dutch men are)

II} the Tibetan agent is Buddhist (as many Tibetan men are)

III} the Tibetan agent is an oriental wise-thinker.

In the case I, the Tibetan agent offers betting quotients $q_1 = 0$ and $q_2 = 1$. However, the Dutch bookie considers the second statement to not be a tautology within his (intuitionistic) logic[50], and the Tibetan agent is told:

> 'Sorry. You've lost! The first statement is a contradiction in any case and so it has $q_1 = 0$. For the second, if we want to be honest[51], I can provide no verification neither for everything is real nor for everything is not; saying that everything is real is true would be a falsity, but also saying that everything is not would be a falsity; therefore I assign $q_2 = 0$ to the second statement[52].
> And I collect your 100€[53] (as I can Dutch-book you).'

The point here (different from the problem of logical omniscience[54]) is that many tautologies of classical logic can no longer be such when considered from another logic. I would call this the problem of logical-system sensitivity.

In the case II the Buddhist[55] Tibetan agent offers betting quotients $q_1 = 1$ and $q_2 = 1$. As he sees the Dutch bookie that is going to collect the money he exclaims:

> 'Sorry! I collect your 100€, not you: those are what Lord Buddha said and

[49] I acknowledge the chat with my professor Hannes Leitgeb for having developed this first point together.

[50] E.g., see (van Dalen 2001).

[51] Within an intuitionistic context, $\neg\neg\mathcal{H}_3 \rightarrow \mathcal{H}_3$ fails, so we cannot obtain the previous equivalence that was valid only in the classical context.

[52] Or, otherwise, if we want to contemplate a suspension of judgment, we could appeal to some kind of Keynes's *principle of indifference*, assigning probability equal to $\frac{1}{2}$, and the same non-classical conclusion would hold.

[53] Here let's suppose that they both agreed on a fixed stake of 100€ on each bet.

[54] Which is very interesting. For example consider '1=1' which is a tautology; but then consider '0+1=1', 'E=1' (where E is a complicated expression equal to 1), 'the odd integer number between 0 and 2 is 1', 'F' (where F is the Fermat conjecture), 'G' (where G is the twin prime conjecture). At what level the proposition is not tautological anymore?

[55] Bodhisattva Nagarjuna's *tetralemma* goes: 'Everything is real and not real; Both real and not real; Neither real nor not real. This is Lord Buddha's teaching.' (Horn 2006)

those are the truths. If you don't know this fact is just because you are ignorant. If you know this fact, but you don't accept the conclusion, it's just because you're too limited mentally and you cannot understand. Anyway that's not my problem: I've won!'

Here, contrarily to the previous case, the Dutch bookie "sees" the two sentences to be one the logical negation of the other one. Nevertheless the Buddhist Tibetan agent doesn't think so.

It could be counter-argued that the reason of what happened comes just from not having the same understanding of the same statements. On the other hand, this is a necessary epistemic problem that arises whenever the assessment of rationality of the group of betting quotients is done in the scenario of a Dutch-book argument: namely, dealing with other people. In other words, how can we be *certain* that we have secured exactly the very same interpretation of language to the agent and to the bookie? Quine's thesis of the *indeterminacy of translation* undermines this certainty, for it's theoretically always possible that there are different accepted interpretations (and therefore translations) to the same sentence (Quine 1987).

Moreover, it's a good chance (due to the "Buddhist" case) to stress the following mismatch: strictly speaking, the Dutch-book argument applies only to formal languages, whereas the subjectivist interpretation of probability wants to give an account of the 'psychological sensation' (which applies all the time also to non-formal languages). Can that strict interpretation of the Dutch-book argument be useful for anything of real life then? There are good reasons to think that, even if it was possible to find a general rigorous formal language for science (which is the most formal among all human discourses), then 'it would not be of much use' (Garber 1983).

In the case `III`, finally, the Tibetan agent reflects long time and utters eventually:

> 'I would think that the first sentence is certainly false and the second sentence is certainly true.. ..but, I must say, life has taught me that one can never be

sure 100% of one's own ideas. And also most of the greatest thinkers of all times came to this conclusion. Therefore I give a $q_1 = 0.01$ and a $q_2 = 0.99$.'

In all the three cases there is disagreement on the probability of the hypotheses. But this ought to be allowed by the subjectivist interpretation[56] of probability as long as any probability judgement reflects one's own 'sensation'; this way for achieving coherence is "not coherent."

On the other hand, in the classical way (i.e., excluding my case with intuitionistic logic) of presenting the Dutch-book argument – including the justification of the axiom of probability 2' – there is the implicit assumption that, that bookie is logically omniscience and that, only one logic is the true objective one, namely classical logic. This assumption, however, contrasts with the subjectivist view of probability, which says that *objective probabilities do not exist*: so, it's not obvious why just a few sentences have an objective probability and all others have not. And, as to me logical objectivism is not an acceptable premise, the Dutch-book argument is not quite sound also due to this.

The bookie's logical omniscience (knowing all logical truths) clearly is an unrealistic idealization. Moreover, by doing so, we also assume a neat divide between logical omniscience and empirical omniscience (knowing all empirical truths)... which, however, has much to do with the weak *analytic-synthetic distinction*. But, supposing to "allow people" to be rationally omniscient in general, then, both the agent and the bookie could always know all the (hypothetically-existing) "objective" values of probability. And, the Dutch-book argument would serve, at most, for preventing from irrational mistakes in assessing betting quotients; but, in this case, we don't need the whole machinery of the Dutch-book argument just for checking our values! Besides, in this case the agent could exactly predict the bookie's future choice and act correspondingly; but, the bookie too could do the same; therefore a paradox. On the other hand, from several sources[57] I've found that the Dutch-book argument is used in a framework of events *simpliciter*, instead of sentences of a formal language, and this implicitly exploits the assumption that the

[56] That is to say, also by the Bayesian one, when the Bayesianism is not objective.
[57] Gillies (2000) included.

bookie is omniscient *simpliciter* (for the axiom 2 of probability would then request him to know all (past and future) certain events).

Finally, would we bet money with someone who *always* pretends to be objectively right in case of disagreement?

Let's *simply* be coherent!

In this section I introduce an original and very simple way (alternative to gambling) for offering coherent measurements of the degrees of belief.

Howson & Urbach write

> 'Ramsey (1931) used the term "consistent" to characterise degrees of belief having the formal stucture of probabilities. De Finetti used the term "coherent", and it is de Finetti's terminology that is usually employed today. We prefer Ramsey's' (Howson & Urbach 1989)

However, I would hardly agree in full with them: the 'terminology [for "coherent"] that is usually employed today' applies to betting quotients of the Dutch book theorem (e.g., see my presentation of the Gillies' proof), but not to degrees of belief. Secondly, "coherent" is not usually defined as having 'the formal structure of probabilities', but as owning a *Dutch-book vulnerability.*
For example, also Hacking (2001) states 'A set of betting rates[58] is coherent if and only if it is not open to a sure-loss contract'.

On the other hand, I think I understand what Howson & Urbach are saying. The idea of coherence or consistency, in fact, is also a general one and can well apply to the degrees of belief too. Hereafter I will use these two terms indistinctly in this more metaphysical sense.

[58]Hacking uses betting rates as a synonym of betting quotient.

In the section ‘Gambling is for the man in the street’ I queried whether we can have inconsistent degrees of belief and I anticipated my doubt about this possibility. Now I just want to “give an image” of my point, by means of the “duck-rabbit” illusion of `Figure 2`, used by Ludwig Wittgenstein (1889-1951) in his *Philosophical investigations*.

`Figure 2`

Here, in fact, I would say that, *at any instant*, *either* we have a full degree of belief in seeing a duck and an empty degree of belief in seeing a rabbit and, *or*, a full degree of belief in seeing a rabbit and an empty degree of belief in seeing a duck, *or*, in the transitions between those two states, we have a partial degree of belief in seeing a duck that is coherently complementary with the partial degree of belief in seeing a rabbit.

Therefore, if a science-fiction ultra-rapid Ramseyan psychogalvanometer were possible, we could not *synchronically*[59] measure degrees of belief that are not “coherent”. Here, as suspectable, the important part is played by the adverb ‘synchronically’. In fact, if the act of measurement takes longer than one instant (i.e., if the degrees of belief are measured *diachronically*[60]), then we could have seen in full both the duck and the rabbit during that interval. Now, clearly, saying ‘I’ve seen the duck and I’ve not seen the duck’ is inconsistent and irrational. Nonetheless, our degrees of belief seem they never were inconsistent or irrational[61].

If the synchronic measurement of the *instantaneous*[62] ‘psychological sensation’ is done *introspectively*[63], then, not easily we would get inconsistent results. Anyway, even an

[59]By synchronically I mean occurring at the same instant.

[60]By diachronically I mean not occurring at the same instant. For example, the measurement could take a few instants just to return one value, which, therefore can be seen as the averaged value over those instants.

[61]There is an old behaviorist tradition to infer the probabilities from the actions we do. Still, a single action can’t result inconsistent *per se*, but only in comparison with other actions carried out at different instants. Moreover, whether there is an irrational inconsistency or not, is rather a relative issue.

[62]By instantaneous I mean considered at a single instant.

[63]I think that the concept of introspection (as a *self*-examination) fits well with the *subjective* interpretation of probability.

introspective synchronic measurement is an operation different from the quantity that we are to measure (which may well be always coherent in a synchronic sense), and so measurements could be biased or even inconsistent. Moreover, also for practical purposes we would then need a rational instrument to allow us to provide measurement that are *temporally representative* of our degrees of belief.

Suppose, as an additional example, we have to quantify *by introspection* our degree of belief for $\mathcal{H}_1$='Italy will win the world cup', and we give $q_1 = 0.9$. Suppose also, that after a minute[64] we are asked to provide a quantification of our degree of belief in $\mathcal{H}_2$='France will win the world cup', and we give $q_2 = 0.2$. Usually there wouldn't be a problem except for the fact that now we realize that France and Italy have just got to play the world-cup final, and, as necessarily one of them has to win, so $q_1 + q_2$ should do exactly 1 apparently. Once we realize this, we feel a certain embarassment because what we have done is *evidence of irrationality.* On the other hand, we really offered an honest and accurate measurement: simply they corresponded to two different times! I'll spend a few more words, in case it's not clear enough.

By introspection we may understand that *at the instant* we gave a $q_1 = 0.9$, we had (more or less implicitly) the sensation of subjective probability for the victory of France measurable as 0.1. Thus, there is no (synchronic) incoherence. Likewise, *at the time* we gave a $q_2 = 0.2$, we had (more or less implicitly) the sensation for the victory of Italy measurable as 0.8. Thus, no (synchronic) incoherence again. Thus no irrationality. Nonetheless, any agent who provided betting quotients like ours, wouldn't look rational!

Therefore I conclude that all seems to suggest that the *instantaneous degrees of belief p are synchronically coherent*; besides, I propose to indicate as $\overleftarrow{q}$[65] the *"averaged" experienced value* of (the measurement of) the subjective probability p in order to give account of what seems to be incoherent: diachronic inconsistencies. I use letter q, as I said at the beginning of the paper, to indicate that that value is a measurement of the degree of belief

[64]But, even if this task was asked without a minute of break, as long as we asked to process them independently, it makes no difference.

[65]The arrow-like symbol of $\overleftarrow{q}$ stands for an "averaged" experienced value (of past), in contrast to $\overrightarrow{q}$ "averaged" expected value (of future), which is well known to mathematics (usually under the simbol $E(q)$). It is another issue to specify on what interval of time the average is made on, but this is ultimately *decided* subjectively, depending on the individual and on the context (. . . see my *Pleasure theory*).

p.

Having observed so, after a little reflection we could "coherently" communicate that our (temporally representative) "probabilities" for Italy and France are $\overleftarrow{q}_1 = 0.85$ and $\overleftarrow{q}_2 = 0.15$ respectively, so that there is no apparent incoherence and we don't look irrational.

In a logical contradiction such as $\mathcal{H}_1 \wedge \neg\mathcal{H}_1$, $\mathcal{H}_1$ can be seen as true (syncronically and diachronically), and $\mathcal{H}_1$ can be seen as false (syncronically and diachronically), but never the mind enters in contradiction synchronically: necessarily $\mathcal{H}_1$ has never $q = 1$ and $q = 0$ at the same time.

So, if synchronic coherence must always occur, then it is of no practical interest: the common interest is not for the instantaneous varying sensation, but for a temporally representative one which is *maximally consistent* (Osherson et al. 1994) with all the instantaneous (varying) sensations[66].

This idea of maximal consistency (among subjective probabilities) is meaningful and the one of interest in cases of diachronic measurements, as in the Italy-France example. In most of practical matter, such as bets, we have to present one representative probabilistic judgment, lasting over time: we cannot say 'No, I've not lost the bet because I thought of that value in only one single instant'.

In the last decades cognitive-psychology research has started studying the processes involved in human probabilistic reasoning, also in a way to make them exploitable for *expert systems* (which need the introduction of human Bayesian priors). That research is trying to develop procedures for correcting incoherence and, thereby, extracting a coherent group of probabilistic evaluations that is *maximally consistent* (e.g., see (Osherson et al. 1994, 2001)). It's worth mentioning (in relation to the thesis of this essay): cognitive psychology uncontroversially holds that degrees of belief exist as a psychological entity, distinct from betting quotient.

The (Bayesian machinery of) Dutch books (or any other sort of gambling) could the-

[66]Interestingly enough, also ethics enters here: we have ethical reasons and motives for appearing as much coherent as possible in front of other people and, presumably, in front of ourself too. This, on the other hand, would prevent a change in opinions and a growth of knowledge.

oretically have worked as a tool for detecting and halting "incoherent" measurements of the degrees of belief, but I hope I have succeeded in convincing that they are rather complicated without flawless benefits.

So, here is my simple proposal: a *wheel of fortune*, like the one of the following figure.

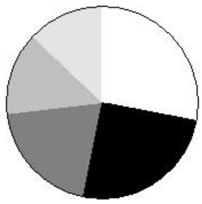

Figure 3

But, what do we do with this "wheel of fortune"?

When we spin this gray-toned wheel of fortune, at least one color, but not more than one, turns out to be the "true" color associated to that spinning. This is analogous to a *partition* of hypotheses[67], i.e., a set mutually exclusive and collectively exhaustive hypotheses (e.g., see (Romeyin 2005) or (Strevens 2006)). That is to say, like in the case of the a gray-toned wheel (full black and full white included), we believe that one of the hypotheses of the partition must be true, but not more than one at the same time.

Then, we take the wheel to have a length of circumference equal to 1 and we associate every different partition to a different gray-toned wheel, and so, every different hypothesis to a different color (tone of gray).

In so doing, the arc of circumference of a specific color has a length that is numerically equivalent to the probability of that specific hypothesis (associated to that color). For example, the wheel of `Figure 3` will correspond to a partition of five hypotheses $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5)$, $\mathcal{H}_1$ being the hypothesis associated to white and, clockwise to, $\mathcal{H}_5$ being associated to lightest gray, so that the related probabilities are $(0.28, 0.25, 0.20, 0.14, 0.13)$.

Therefore, as before I concluded that *instantaneous* degrees of belief are synchronically *coherent*, then they act over a *partition* of hypotheses and they can be simply represented by a *wheel* of fortune. Moreover, it visually follows that the instantaneous degrees of belief

[67]Roughly speaking a partition is a complete set of complementary hypotheses of the same type: e.g., Italy wins or France wins (the final). But Italy wins or France or I see a duck is not a partition for Italy wins and I see a duck can be both true.

are a probability function, as our wheel of fortune clearly *is* a *probability measure*.

Please note that here I've brought some argument to support the claim that degrees of belief are mathematically describable as a probability, while this was assumed *a priori* in the Dutch-book argument and no sound justification has come from that argument[68]. The Dutch-book argument is not clear about this point because it aims to show that the degrees of belief of a rational agent are probability functions, therefore it implicitly assumes that they can be otherwise (by an irrational agent). In this paper I've sustained the de Finetti's 'psychological sensation' as a founding definition of degrees of probability.

In the usual context of the Dutch-book argument, irrationality would follow from incoherence, but, not in the context advanced here: in fact it's rather difficult to conceive of how a psychological sensation con be irrational *per se*.

Up to now, however, I've just just showed that instantaneous degrees of belief are coherent and therefore they satisfy the axioms of probability. Nevertheless, too frequently we provide subjective probabilities that not coherent and therefore irrational. I've sustained that those values mainly correspond to diachronic measurements of degrees of belief. However, I have not yet provided an alternative to flush out such irrationalities.

How to flush out the irrational values?

My proposal would simply consist in choosing the wheel of fortune that best fits the "averaged" experienced value of the degree of belief! If we do so, then we *necessarily* provide values that are probabilistically coherent (they add up to 1, etc.) and this is nothing, but what we want[69].

If we have accepted that the degrees of belief are probability functions, then, what we would find (rationally) representative of them must be a probability function too.

On the other hand, how to help to "facilitate" our measure?

Maher used the striking scenario 'the agent is told she will be shot if her betting quotients

[68]For example, about the axiom 2 of probability, the Dutch-book argument aims to show nothing but that the values of subjective probability we rationally assign to tautologies must be equal to 1. . . . But this is right what the definition of probability imposes!

[69]At least from a purely subjectivist perspective. But also objectivist perspectives must satisfy subjectivist ones first.

don't satisfy the axioms of probability' to exemplify the major fallacy of the Dutch-book argument. I offer a somehow similar counterfactual scenario for my alternative approach: suppose that a completely-trustworthy god that "reads" our degrees belief; then this god tells us that the more representative we measure our degree of belief and the less we will experience the hell.

Here we don't commit the fallacy to try to justify (badly) what we have already assumed: we already know that we want something that works like a probability function. By choosing a certain wheel more than another one we already have this fulfilled. On the other hand, let's just try to make our measurement authentic as much as possible, in order to make it really *even more rational*.

The thought-experiment with the omniscient and completely-trustworthy god can be a good motivation to do the assessment. On the other hand, if one prefers, we can think about other more-realistic cases. Anyway this has not to be a conceptual obstacle: let's suppose that we do have some motivation for rationally carrying out the numerical assessment. The request of rationality is more fulfilled just by the very idea of wheel of fortune: (as it is a rational constrant and) as it is a very intuitive idea, the management of the transparency of the measurement is more reliable. In addition when we have in mind a certain wheel of fortune we "see all the probabilities at the same time" (and this is quite close to be a synchronic measurement).

Moreover, this idea plays also a direct *link between subjective and objective probability*: the psychological sensation and the physical length of the arc of wheel. For example about the certainty of an $\mathcal{H}_j$, an agent is requested to judge it certain only if the strength of her sensation equal the one she has for the case where a full-white wheel is likely to give a white color.[70]

Besides, this "help" to measure the degrees of belief is carried out just subjectively and without (potential or actual) Dutch bookie. Not negligibly, even a Dutch bookie is a per-

[70]In my personal work I define the *DDB* *d*esciptive *d*egree of *b*elief of something as the psychological/subjective estimate of the proportion of the physical/objective wheel that represents its chance to occur. It could be expressed by taking a white disk of paper and coloring with the black proportion of it, and with other colors the proportion of other data/theories, so that coherence in necessarily fulfilled. For the case of a continuos distribution, a distribution could be drawn over the interval from 0 to 1, and then a device could normalize it and paint in gray tones on the white disk.

son like us, and *he too* acts in order to not lose money, and *he too* could do a mistake! It's also important to say that not few philosophers (e.g., Hájek, Howson, Urbach) construe the betting quotients as which an agent would consider fair *without* of a possible actual testability by gambling. But then, without abusing of improper idealizations (i.e., impossible to a real gambling scenario), why not just to take that one has some motivation in assigning our "best" values of probabilities *with* the actual testability offered by a wheel of fortune[71]? And, obviously, the best values of probability (like for the outcome of a wheel from a spinning) must obey the axioms of probability.

In this section I have outlined the need for setting up a theoretical differentiation of the subjective probability in a descriptive one (for the instantaneous degrees of belief) and in a normative one (for the "averaged" experienced value of the degrees of belief we ought to have)[72]. The psychological sensations that we have and the one that what we ought to have are different conceptual entities and they undergo different theoretical labor. Bayesianism does not do such an analysis, even though the awareness of the necessity to make a distinction of this kind between, (and to create) two different concepts of Bayesian probability, is increasing[73].

Last, but not least, all my 24 objections raised against the Dutch-book argument would vanish in front of the wheels of fortune (including the ones related to the axiom 2 and 2' of probability).

Some philosophers argue that the figure of the cunning bookie has to be suitably construed for it involves an element of idealization. I simply think that such an idealization is bad.[74]

[71] Which is: between 0 and 1, adding up to 1 for any partition. In this way, the service offered by a wheel (or more in case of more than 1 partition) is not less than the one of a cunning bookie: satisfying rules like the special and general addition rules for disjunctions, the special and general multiplication rules for conjunctions, the implication and equivalence rules, without troubles for the axiom concerning necessary truths.

[72] In my works they are respectively named descriptive degree of certainty *DDC*, and *PDC* *p*rescriptive *d*egree of *c*ertainty.

[73] (Hawthorne forthcoming).

[74] Finally I would like just to give a sketch of idea about how my wheels would be a clearer support (and so a useful support) for cases related to infinities of hypotheses.
In fact, as the `Figure 4` shows, we can represent infinite hypotheses: in that wheel there are infinite tones of gray.
Moreover the same wheel has also one hypothesis (the white) whose probability is *non*-infinitesimal, but finite. This would correspond to the mathematical concept of mixed-type random variable, which is a possible unification of discrete and continuous random variable (which can have associated also a continuous *non-uniform* random variable), but it's visually quite simple now.

Then we can deal immediately with an important problem of finite/countable additivity (axiom 3' of probability): de Finetti offered the simple problem 'Guess a number $n \in \mathbb{N}$ chosen*at random*'. Let's suppose that $\mathcal{H}_1$ is the hypothesis that the chosen number is (the first number) 1, and $\mathcal{H}_i$ is the i-th number. Clearly the set of the (natural) numbers forms a partition and so $p(\mathcal{H}_1 \vee \mathcal{H}_2 \vee \ldots \vee \mathcal{H}_i \vee \ldots) = 1$. On the other hand, here, he refused to accept the relation (countable additivity):

$$p(\mathcal{H}_1 \vee \mathcal{H}_2 \vee \ldots \vee \mathcal{H}_i \vee \ldots) = p(\mathcal{H}_1) + p(\mathcal{H}_2) + \ldots + p(\mathcal{H}_i) + \ldots$$

for the following argument. As there are infinite numbers, we should be able to make a uniform distribution of probability over this set, then any number has the same probability equal $p(\mathcal{H}_i) = 0$ (given by dividing the probability of the partition (1) by the number of equiprobable elements (∞)).
Therefore the right-hand-side sum of the previous relation must be zero, which is different than 1!

I don't accept the de Finetti's argument for the following reason: the division of a unity by an infinity does an infinitesimal quantity.
I discussed this point with Jon Williamson and he told me that even if we reason by infinitesimals we end up with a paradox: if $p(\mathcal{H}_i) = \varepsilon > 0$ then $\sum_i p(\mathcal{H}_i) = \infty$!
I don't accept this argument either: in fact when carry out the calculus of an integral, we sum over an infinity of infinitesimal quantities, but the sum doesn't necessarily go to infinity.
My solution to the problem would be to substitute the sum in axiom 3' (valid only for discrete distribution) by a general "sum operator". The idea of this is present in my *Modeling of a Simple Physical World*, but I will explain it better in a more suitable place than here.

But, I try to express my idea in images. Let's now consider the problem: 'Guess a color chosen *at random* of the wheel of `Figure 5`'. On that wheel there are infinite colors, distributed in a uniform way. It is obvious that any color has an infinitesimal probability and the sum (of probability) of all colors does 1: the whole wheel.

Can we do the same with numbers?
Well, my answer is: if numbers belong to a limited set (e.g., the limited real interval $[0,1] \in \mathbb{R}$), then we can. Otherwise we cannot. ... My argument?
First, let's call the top highest point of the wheel as 0. Starting from it clockwise, we then measure the arc length $s \in [0,1]$. The bottom of the wheel, therefore will have arc length equal to $\frac{1}{2}$.
Let's suppose we uniformly distribute 100 integers along the wheel, then the bottom will have the number 50.
Let's suppose we uniformly distribute 10000 integers along the wheel, then the bottom will have the number 5000.
On the other hand, when we uniformly distribute infinite integers along the wheel, then the bottom must have an infinite value (otherwise the left top will be valued just finitely double than the bottom, and not infinity).
Therefore, as we can apply the same argument by any finite arc length (except for $s = 0$ corresponding to $i = 1$ obviously), then it follows that it's not possible to assign a uniform distribution of probability over an unlimited set like the natural numbers.

Does, on the other hand, countable additivity hold without the hypothesis of uniformity?
Well, my answer is: yes. And I can give also a visual example by a wheel of fortune!
Let's suppose, as it is reasonable, that the number is chosen like: the greater it is, the less probable it gets.
Then, as an example, we can represent the infinite numbers and also their probabilities by means of

$$i = \left\lfloor \frac{1}{1-s} \right\rfloor$$

whereas $\left\lfloor \frac{1}{1-s} \right\rfloor$ is the floor function of $\frac{1}{1-s}$, also called *integer value* of $\frac{1}{1-s}$, and by means of the wheel of fortune of `Figure 6`.

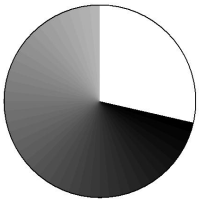

Figure 4

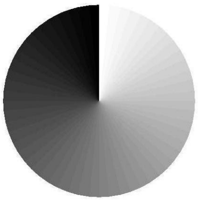

Figure 5

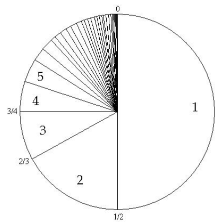

Figure 6

Conclusion

Gambling behavior[75] has been traditionally used to define the concept of subjective probability.

I've defended the claim that gambling is unnecessary as a means of defining the subjective probability.

The (synchronic) Dutch-book argument is usually brought as a justification of the fact that our degrees of belief obey the probability calculus and it currently is the most compelling link between rational degrees of belief and mathematical probability.

I've brought 24 objections against the Dutch-book argument and showed that the alledged link is unsuitable.

I have offered an extremely-intuitive alternative to soundly quantify the degrees of belief in a way that they necessarily turn out to be coherent in the classical probabilistic sense.

In conclusion, (at least) when considering probability, gambling is not just inessential, but

[75]The link between probability and behavior is present in my *Pleasure Theory* (which is a theory of decision-action) where the p there exactly correspond to the descriptive degree of certainty DDC in this essay (and not q, nor the normative PDC).

it's just bad.

Bibliography

Bartha, P 2004, 'Countable additivity and the de Finetti lottery', *British Journal for the Philosophy of Science*, vol. 55, pp. 301-321.

Bradley, D & Leitgeb, H 2006, 'When betting odds and credences come apart: more worries for Dutch book arguments', *Analysis*, vol. 66, no. 2, pp. 119-27.

van Dalen, D 2001, 'Intuitionistic logic', in L Gouble (ed.), *The Blackwell guide to philosophical logic*, Blackwell, Oxford.

Eagle, A 2006, 'Betting on one's own death', draft, Oxford University.

Earman, J 1992, *Bayes or bust? A critical examination of Bayesian confirmation theory*, MIT Press, Cambridge MA.

Galavotti, MC 2001, 'Subjectivism, objectivism and objectivity in Bruno de Finetti's Bayesianism', in D Corfield & J Williamson (eds), *Foundations of Bayesianism*, Kluwer, Dordrecht.

Galavotti, MC 2003, 'Kinds of probabilism', in P Parrini, WC Salmon & MH Salmon (eds) *Logical empiricism: historical and contemporary perspectives*, The University of Pittsburgh Press, Pittsburgh.

Garber, D 1983, 'Old evidence and logical omniscience in Bayesian confirmation theory', in J Earman (ed.), *Testing scientific theories*, University of Minnesota Press, Minneapolis.

Gillies, D 2000, *Philosophical theories of probability*, Routledge, New York.

Glymour, C 1998, 'Why I am not a Bayesian', in M Curd & JA Cover (eds), *Philosophy of science*, W.W. Norton & Company, New York.

Hacking, I 2001, *An introduction to probability and inductive logic*, Cambridge University Press, Cambridge UK.

Hájek, A 2002, 'Interpretations of Probability', in EN Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, viewed 03 November 2005, <http://plato.stanford.edu/archives/sum2003/entries/probability-interpret>.

Hájek, A 2005, 'Scotching Dutch books?', *Philosophical Perspectives*, vol. 19, no. 1, pp. 139-151.

Hájek, A & Ericksson, L forthcoming, 'What are degrees of belief?', *Studia Logica*.

Hanson, SO 2003, 'Ten philosophical problems in belief revision', *Journal of Logic Computation*, vol. 13, no, 1, pp. 37-49.

Hawthorne, J forthcoming, 'Degree-of-belief and degree-of-support: why Bayesians need both notions', *Mind*.

Heilig, K 1978, 'Carnap and de Finetti on bets and the probability of singular events: the Dutch book argument reconsidered', *British Journal for the Philosophy of Science*, vol. 29, pp. 325-346.

Horn, LR 2006, ‘Contradiction’, in EN Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, viewed 04 April 2007, <http://plato.stanford.edu/entries/contradiction>.

Howson, C & Urbach, P 1989, *Scientific reasoning: the Bayesian approach*, Open Court, La Salle.

Huber, F 2006, ‘Belief and degrees of belief’, draft, California Institute of Technology.

Jackson, F & Pargetter, R 1975, ‘A modified Dutch book argument’, *Philosophical Studies*, vol. 29, pp. 403-407.

Jeffrey, RC 2004, *Subjective probability: the real thing*, Cambridge University Press, Cambridge UK.

Lewicki, MS 2001, *Bayesian inference*, lecture slides, Carnegie Mellon University, Pittsburgh.

Maher, P 1993, *Betting on theories*, Cambridge University Press, Cambridge UK.

Nau, RF 2001, ‘De Finetti was right: probability does not exist’, *Theory and Decision*, vol. 51, pp. 89-124.

Osherson, D, Shafir, E & Smith, EE 1994, ‘Extracting the coherent core of human probability judgement: a research program for cognitive psychology’, *Cognition*, vol. 50, pp. 299-313.

Osherson, D, Lane, D, Hartley, P & Batsell, RR 2001, ‘Coherent probability from incoherent judgment’, *Journal of Experimental Psychology*, vol. 7, no. 1, pp. 3-12.

Quine, WV 1987, ‘The indeterminacy of translation again’, *The Journal of Philosophy*, vol. 84, no. 1, pp. 5-10.

Romeyn, J-W 2005, *Bayesian inductive logic*, PhD thesis, University of Groningen, Groningen.

Rowbottom, DP forthcoming, ‘The insufficiency of the Dutch book argument’, *Studia Logica.*

Strevens, M 2006, *Notes on Bayesian confirmation theory*, viewed 15 April 2007, <http://www.nyu.edu/gsas/dept/philo/user/strevens/Classes/Conf06/BCT.pdf>.

van Fraassen, BC 1989, *Laws and symmetry*, Oxford University Press, Oxford.

Wakker, PP 2001, *History of the term Dutch book*, viewed 17 April 2007, <http://www.brunodefinetti.it/Bibliografia/History of the Term Dutch Book.htm>.

Williamson, J 1999, ‘Countable additivity and subjective probability’, *British Journal for the Philosophy of Science*, vol. 50, no. 3, pp. 401-416.

Williamson, J 2005, *Bayesian nets and causality: philosophical and computational foundations*, Oxford University Press, Oxford.

Zoccante, S 2005, *Fondamenti storici ed epistemologici della probabilità*, viewed 17 April 2007, <http://www.math.unipd.it/˜azanardo/ssis/zoccante.pdf>.